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A new generalized proximal-point method for convex optimization problems in Banach spaces.

Vikas Rajput* & Alok Kumar Verma**
*P.G Department of Mathematics & Computer Applications, Ch. Sughar Singh Educational Academy, Jaswantnagar, Etawah
**P.G Department of Mathematical Science and Computer Applications, Bundelkhand University, Jhansi (U.P), India.
Email – **rajvasu99@gmail.com*

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Abstract

In this paper we aim to provide that the extended the area of applicability of the generalized proximal point method to solving convex optimization problems in Banach spaces. There are several novel aspects. This monograph includes and closely related Tikhonov regularization method for convex optimization problems. Our discussion mainly based on previously research article and fixed point theory. Our results are mainly depends on previous study and only changing domain and co-domain which may extended area for finding solution of convex optimization problems and also work on some examples to justify our results and conclusion. We will see that proximal operators and proximal algorithms have a number of interesting interpretations and are connected to many different topics in optimization and applied mathematics.

1. Introduction

The books name 'Convexity and optimization in Banach spaces' provides a self-contained presentation of basic results of the theory of convex sets and function in infinite-dimentional spaces. The main emphasis is on applications to convex optimization and convex optimal control problems in Banach spaces [1]. There are three important basic areas of non-linear analysis, convex analysis, monotone operator theory and the fixed point theory of non expansive mappings [2].

New generalized proximal point method

Let B be a Banach space and $g: B \to \mathbb{R}$ a continuity convex function. Now consider the optimization problem

$$\min_{x \in B} g(x) \quad (1)$$

We denote by S ⊆ B be the set of solutions which is not empty for finding optimization solution and $g_{min} \in (0,1]$ then S be the infimum of g(x).

Optimization problem (1) for a given λ> 0 so we consider the proximal point mapping

$$prox_{\gamma g}(\text{u}) = \underset{x \epsilon B}{argmin} \left\{ g(x) + \frac{1}{2\lambda} \|x - u\|^2 \right\} \quad (2)$$

Note that the right- hand side is hold convexity so mean of recurrence

$$x^{n+1} = prox_{\lambda_n g}(x^n) \quad (3)$$

Now By equation (2) & (3)

$$x^{n+1} = \underset{x \epsilon B}{argmin} \left\{ g(x) + \frac{1}{2\lambda} \|x - u\|^2 \right\} = prox_{\lambda_n g}(\text{u}) \quad (4)$$

Here $\lambda_n > 0$ is perimeter, x^n ⊆B be an arbitrary bounded sequence of modified proximal point method and Tikhonov method. $x^n$is sequence and bounded so all the adherent point are solution of convex optimization problem.

There are four term for the convex optimization problems which are –

(i) If n=1 then x^1 ∈ B, M ⊆ B is bounded set

(ii) If x^n is a sequence so all adherent point of this sequence are solution of the optimization Problem (1)

(iii) We takeu^n ∈ M, $\lambda_n >$ 0, and let $x^{n+1} := prox_{\lambda_n g}(u^n)$, i.e. x^{n+1} solve

$$\min_{x \in B} g(x) + \lambda_n \|x - x^n\|^2$$

(iv) Iterate n→ $n+1$, and so on then first we take u^n by x^n and the iteration method apply to find all adherent point of sequence. The set of all adherent point is the solutions of convexoptimization problem.

Theorem -1:-

Let (x^n) be the sequence generated by the four term, and x ∈ B. Then

$$g(x^{n+1}) - g(x) \leq \lambda_n \{\|x - u^n\|^2 - \|x^{n+1} - u^n\|^2\} \quad (5)$$

$$\leq \lambda_n \|x - u^n\|^2 \quad (6)$$

For every n ≥ 1. Furthermore, if(x^n) is bounded. Which gives S is not empty.

Proof:

The given that by definition of sequences x^{n+1} implies that

$$g(x^{n+1}) + \lambda_n \|x^{n+1} - u^n\|^2 \leq g(x) + \lambda_n \|x - u^n\|^2 \forall\, n \geq 1$$

Since$x = x^* \in C$this gives$g(x^*) \leq g(x^{n+1})$, by the inequality of equation (5) and this quantity is non negative.

Now from this obtain $\|x^{n+1} - x^n\| \leq \|x^* - x^n\|$ for all n ≥ 1. This gives(x^n) is bounded then u^n is also bounded.

Theorem- 2:-

From theorem-1 Let (x^n) be generated by term, and let $\lambda_n \to 0$. Then$g(x^n) \to g_{min}$ and every adherent point of (x^n) is a solution of the optimization problem (1) and$x^* \in S$, then S is not empty

$$g(x^{n+1}) - g_{min} = \varphi(\lambda_n \|x^{n+1} - x^*\|) = \varphi(\lambda_n)$$

Proof:-

by the claim $\lambda_n \to 0$, u^n is bounded and the inequality (6)

$$\lim_{n\to\infty} \inf g(x^{n+1}) \leq g(x) \ \forall x \in B.$$
$$\Rightarrow g(x^{n+1}) \to g_{min}$$

By the theorem-1 continuity of $g(x)$ and the theorem-2 $x = x^* \in S$, boundedness of x^n then sequence of function values $g(x^n)$. If S is empty then x^n is unbounded which is contradiction from our assumption so hence x^n is bounded and all adherent point are solution of equation (1).

Theorem-3:-

Let g be convex optimization and let $\lambda_n \to 0$. Then (x^n) converges to the unique element $x^* \in S$ so
$\|x^{n+1} - x^*\| = \varphi(\lambda_n)$ and $g(x^{n+1}) - g(x^*) = \varphi(\lambda_n{}^2)$

Proof:-

From convexity there exist constant $a > 0$ such that

$$a\|x^n - x^*\|^2 \leq g(x^n) + g(x^*) - g(x^n + x^*)$$

By theorem-3 $x^n \to x^*$ then

$$\leq g(x^n) + g(x^*) - 2g(x^*) = g(x^n) - g(x^*)$$

Hence by the iteration method

$$g(x^{n+1}) - g(x^*) = \varphi(\lambda_n \|x^{n+1} - x^*\|)$$

Now induction process, $a_0 > 0$ such that

$$a_0\|x^{n+1} - x^*\|^2 \leq (g(x^{n+1} - gx^*)) \leq a_1\|x^{n+1} - x^*\|\lambda_n$$

Which gives the completeness set of equation (1) so hence S is the solution set of equation (1) and closed & convexity.

Theorem 4:

Let (x^n) be generated sequence, let $\lambda_n \to 0$, and let $u^n \to u^*$ for some $u^* \in B$. If S is not empty, then

$$x^n \to P_C(u^*)$$

Proof:-

If $g(x^*) \leq g(x^{n+1})$ and $x^* \in S$ so x^* an arbitrary solution because S is not empty. From equation (5)

$$\|x^{n+1} - u^n\| \leq \|x^* - u^n\| \ \forall x^* \in S$$

and all n ≥ 1, u^n is also sequence then

$$\lim_{n\to\infty} \inf\|x^{n+1} - u^*| \leq \lim_{n\to\infty} \inf(\|x^{n+1} - u^n\| + \|u^n - u^*\|)$$
$$= \lim_{n\to\infty} \inf\|x^{n+1} - u^n\|$$
$$\leq \lim_{n\to\infty} \inf\|x^* - u^k\|$$

$$= \|x^* - u^*\|.$$

By the theorem-1 and Theorem-2, there is belongs a subset of all adherent point which is one K $\subseteq$ $\mathbb{N}$such that $x^{n+1} \to Kx$ for some $x \in S$. Now x^* is a particular solution $x^* := P_S(u^*)$, then

$$\|x - u^*\| \geq \lim_{n\to k} \inf \|x^{n+1} - u^*\| \geq \|x^* - u^*\|,$$

If x is a solution of generating sequence and x^* is also solution of this sequence x^n for optimization problem hence, $x = x^*$ so

$$\|x^{n+1} - u^*\| \to K\|x - u^*\|,$$
$$\Rightarrow x^{n+1} \to Kx = x^*$$

Thus $x^n \to x^*$, these result gives us iterates generated sequence converge to the minimum norm solution of equation (1) and if S is not empty. If generating sequence x^n having only one adherent point or more.

Examples and counter examples

We now try give some examples which illustrate the convergence assertions of the theorems. The convergence results can give best optimize solution. We know that convex optimization problem is to find best solution so uses these theorems we try to finding optimization solution by generated sequence.

Examples 1: -Consider the convex function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) := max\{0, x^2 - 1\}$ find optimization solution in Banach space and interval (0,1] ?

Explanation:-

The given convex function is

$$g(x) = \max\{0, x^2 - 1\}$$
(a)

By equation (a) the generated sequence and gives all adherent points are

$$x^2 - 1 = 0$$
$$\Rightarrow x = 1, -1$$

The solution set of the corresponding optimization problem (1) is given by $S = [-1,1]$. Now, by four term theory with $\lambda_n = n$ and the alternating sequence $u^{2n} = 1, u^{2n+1} = -1$ for all n. Then x^{2n+1} is the unique solution of

$$\min_{x} max\{0, x^2 - 1\} + 4n(x - 1)^2$$

Which is given by $x^{2n+1} = 1$since this number minimizes both terms separately, whereas $x^{2n}$is the solution of

$$\min_{x} max\{0, x^2 - 1\} + 2(2n - 1)(x + 1)^2$$

Therefore given by $x^{2n} = -1$for similar reasons. Hence, we eventually get the alternating sequence $(-1,1,-1,1 \dots .)$. There are two adherent point 1 and -1 but only $1 \in (0,1]$ and by convexity property adherent point 1 is optimization solution of given convex function.

Example 2:-

Let $g: \mathbb{R} \to \mathbb{R}$ be the function given by

$$g(x) = \begin{cases} x^{\alpha} & if\ x > o \\ 0 & otherwise \end{cases}$$

For some constant $\alpha > 0$. Find solution in Banach spaces and semi interval$]0,1[$?

Explanation: Hence$g_{min} = 0$, but the maximum is not attained. If $(u^n) = 0$ for all n, then x^{n+1} is the maximum of $x^{-\alpha} + x^2/2\lambda_non]0,1]$. We uses theorem-1 for generating sequence and by the iteration $x^{n+1} = (\alpha\lambda_n)^{1/\alpha+2}$ and

hence,

$g(x^{n+1}) = \varphi(1/\lambda_n{}^d)$With d= $\alpha/(\alpha + 2)$.

This given problem is not generating convergence sequence so hence there is not exist adherent point so is not convex.

Example 3:-

The mapping $g: \mathbb{R} \to \mathbb{R}$, and the function is $g(x) = x^2$. Find optimization solution in $(0,1]$?

Explanation:-

The given convex function and using theorem then

$$x^2 = 0 \Rightarrow x = 0$$

So by theorem-1 the generated sequence which adherent point is 0. Then $u_n = 1/\lambda$ if $\lambda \to \infty$ hence u_n has only one adherent point which is 0 but $0 \notin (0,1]$ so this adherent point is not optimization problem.

Example 4:-

Let $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 1/\log(x), x > 0$.Find the optimal solution in the semi interval $(0,1]$?

Explanation:-

The given function $g(x) = 1/\log x$(b)

By using theorem-1 and theorem-2 the generating sequence is $u_n = x^{-1}$ when $x \to \infty$ then the adherent point is only one that is $0 \notin (0,1]$ hence there exist generating sequence but the equation (b) is non convex.

2. Conclusion& Discussion

We have proved that new method of generalization the well-known proximal-point method for convex optimization problem in Banach spaces with definite interval$(0,1]$and extended field for generating sequence, find its adherent point which are solutions of the convex problem. This iterated process based on a generalized proximal point method for convex optimization problems in Hilbert spaces, Tikhonov regularization method for convex optimization problems. Our simply discussion is that method of generalization of the well-known proximal-point with change perimeter and interval.

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References

[1] Barbu V, Precupanu T. Convexity and optimization in Banach spaces. 4th ed., Springer monographs in Mathematics. Dordrecht: Springer; 2012.

[2] Bauschke HH, Combettes PL. Convex analysis and monotone operator theory in Hilbert Spaces. New York (NY):Springer; 2011. (CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC).

[3] Güler O. On the convergence of the proximal point algorithm for convex minimization. SIAM J Control Optim.1991;29(2):403–419.

[4] Martinet B. Régularisation d'inéquations variationnelles par approximations successives. Rev Française Informat Recherche . 1970;4(Ser. R-3):154–158.

[5] Martinet B. Détermination approchée d'un point fixe d'une application pseudo-contractante. Cas de l'application prox. C R AcadSci Paris Sér A-B. 1972;274:A163–A165.

[6] Rockafellar RT.Monotone operators and the proximal point algorithm. SIAM J Control Optim.1976;14(5):877–898.

[7] Engl HW, Hanke M, Neubauer A. Regularization of inverse problems. Dordrecht: Kluwer Academic Publishers Group; 1996.

[8] Tikhonov AN, Morozov VA. Methods for the regularization of ill-posed problems. VychislMetodyiProgrammirovanie.1981;35: 3–34.

[9] Bertsekas DP. Constrained optimization and Lagrange multiplier methods. New York-London: Academic Press [Harcourt Brace Jovanovich, Publishers]; 1982.

[10] Iusem AN. Augmented Lagrangian methods and proximal point methods for convex optimization. Investigaci'on Operativa. 1999; 11–50.

[11] Rockafellar RT. A dual approach to solving nonlinear programming problems by unconstrained Optimization. Math Program.1973;5:354–373.

[12] Christian Kanzow, Daniel Steck. A generalized proximal point method for convex optimization problems in Hilbert spaces. Taylor & Francis 2017.

[13] Birgin EG, Martínez JM. Practical augmented Lagrangian methods for constrained optimization. Philadelphia (PA): SIAM; 2014.

[14] Andreani R, Birgin EG, Martínez JM, SchuverdtML. On augmented Lagrangian methods general lower-level constraints. SIAM J Optim.2007;18(4):1286–1309.

[15] Birgin EG, Floudas CA, Martínez JM. Global minimization using an augmented Lagrangian method with variable lower-level constraints. MathProgram.2010; 125(1, Ser. A):139–162.

[16] Andreani R, Birgin EG, Martínez JM, Schuverdt ML. Augmented Lagrangian methods under the constant positive linear dependence constraint ualification. Math Program.2008;111(1–2, Ser. B):5–32.

[17] Kanzows C. On the multiplier-penalty-approach for quasi-variational inequalities. Math Program.2016;160(1–2,Ser. A):33–63.

[18] Kanzow C, Steck D. Augmented Lagrangian and exact penalty methods for quasi-variational inequalities. Technical Report. Institute of Mathematics, University of Wurzburg; June 2016.

[19] Pang J-S, Fukushima M. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. Comput Manage Sci. 2005;2(1):21–56.

[20] Kanzow C, Steck D. Augmented Lagrangian methods for the solution of generalized Nash equilibrium problems. SIAM J Optim.2016;26(4):2034–2058.

[21] Kanzow C, Steck D, Wachsmuth D. An augmented Lagrangian method for optimization Problems in Banach spaces. Technical Report. Institute of Mathematics, University of Wurzburg; December 2016.

[22] Kanzow C, SteckD. An example comparing the standard and modified augmented Lagrangian methods. Technical Report. Institute of Mathematics, University of Wurzburg; February 2017.

[23] Güler O. New proximal point algorithms for convex minimization. SIAM J Optim.1992; 2(4):649–664.

[24] A.N Iusem, D. Butnariu .On a proximal-point method for convex optimization in Banach spaces. Taylor & Francis, 2007.

[25] R.S. Burachik. Generalized proximal point methods for the variational inequality problem, Ph.D. Thesis, Institute de Mathematica Pura e Aplicada, Rio de Janerio,1995.

[26] A. Ya., R.S. Burachik and A.N. Iusem. A proximal point method for nonsmooth convex optimization problems in Banach spaces.

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